Nonclassical Potential Symmetries and New Explicit Solutions of the Burgers Equation

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Several new nonclassical potential symmetry generators to the Burgers equation are derived. Some explicit solutions, which cannot be derived from the Lie symmetry group of Burgers or its adjoined equation, are obtained by using these nonclassical potential symmetry generators.

Key words: Nonclassical Potential Symmetry; Explicit Solution; Burgers Equation.

1. Introduction

The symmetry group method is important and widely used in the reduction and construction of explicit solutions of PDEs (partial differential equations). As shown in [1 – 4], the Lie symmetry group method can be used to find explicit solutions of PDEs using symmetry reduction and construction. This method is known as the classical method. By a classical symmetry group of a system of PDEs we mean a continuous group of transformations, which acts on the space of independent and dependent variables and transforms one solution of PDEs into another solution. These solutions are called group-invariant solutions.

The first approach to potential symmetries of a system of PDEs was made by Bluman and Cole [5]. In [6], Bluman and Kumei introduced an algorithm which yields new classes of symmetries of given PDEs which are neither Lie point nor Lie-Bäcklund symmetries. They are nonlocal symmetries and are called potential symmetries. In general, the number of determining equations in this kind of symmetry is smaller than in the classical Lie group method. Therefore it is difficult to find all possible solutions of the overdetermined system. Using this new symmetry method, a much wider class of symmetry groups is available. Hence there is the possibility of finding more group-invariant solutions by the same reduction technique. This new symmetry attracts many researchers.

Recently, the potential symmetry has been developed and generalized to nonclassical potential symmetries. Several nonclassical potential symmetry generators of the wave equation are obtained in [7], but not any new solution, different from the ones obtained by using the Lie group method, is derived. A few new group-invariant solutions of the Burgers equation have been obtained in [8] by using its nonclassical potential symmetry group. The Burgers equation has been studied by many researchers, and many explicit solutions and interesting results have been obtained by using different methods (see [9 – 13] and references cited therein).

In this paper, new nonclassical potential symmetry group generators of the Burgers equation are obtained. Several previously unknown explicit solutions of the Burgers equation are obtained by applying the nonclassical Lie group method to these new symmetry generators.

2. Symmetries and Solutions

In this section we study the nonclassical potential symmetry and the invariant solutions of the Burgers equation for a field $u(x,t)$ of the form

$$u_t + uu_x - u_{xx} = 0. \tag{1}$$

Associated to (1) are the equations

$$\begin{aligned} v_x &= u, \\ v_t &= u_x - \frac{u^2}{2}. \end{aligned} \tag{2}$$

If we substitute the first equation of (2) into the second one, we obtain the adjoint equation

$$v_t + \frac{v_x^2}{2} - v_{xx} = 0. \quad (3)$$

In this paper we use nonclassical potential symmetries of the system (2) to seek invariant solutions of (1). To this end, let

$$V = \tau(x,t,v)\frac{\partial}{\partial t} + \xi(x,t,v)\frac{\partial}{\partial x} + \phi(x,t,u,v)\frac{\partial}{\partial u} + \eta(x,t,v)\frac{\partial}{\partial v} \quad (4)$$

be an infinitesimal generator depending on the independent variables x,t and on the dependent variables u,v. We have to determine all possible coefficient functions $\tau, \xi, \phi,$ and η. The corresponding one-parameter group $\exp(\varepsilon V)$ is a symmetry group of the system (2). According to the classical Lie symmetry group theory, the symmetry determining equation of the system (2) is

$$\begin{aligned} &\eta_x + v_x\eta_v - v_t(\tau_x + v_x\tau_v) - v_x(\xi_x + v_x\xi_v) = \phi, \\ &\eta_t + v_t\eta_v - v_t(\tau_t + v_t\tau_v) - v_x(\xi_t + v_t\xi_v) = \\ &\phi_x + u_x\phi_u + v_x\phi_v - u_t(\tau_x + v_x\tau_v) - u_x(\xi_x + v_x\xi_v) - u\phi. \end{aligned} \quad (5)$$

Combining (1), the associated system (2), and the adjoint equation (3), we obtain

$$\begin{aligned} u_x &= \frac{u^2}{2} + \frac{1}{\tau}(\eta - u\xi), \\ u_t &= \frac{\phi}{\tau} - \frac{\xi}{\tau}\Big[\frac{u^2}{2} + \frac{1}{\tau}(\eta - u\xi)\Big]. \end{aligned} \quad (6)$$

Substituting now (6) into the first equation of (5), we obtain

$$\phi = \eta_x - \frac{\eta}{\tau}\tau_x + u\Big(\eta_v - \xi_x + \frac{\xi}{\tau}\tau_x - \frac{\eta}{\tau}\tau_v\Big) + u^2\Big(\frac{\xi}{\tau}\tau_v - \xi_v\Big). \quad (7)$$

Substituting now (6) and (7) into the second equation of (5), we get

$$\begin{aligned} &\eta_t + \frac{\eta}{\tau}(\eta_v - \tau_t) - \frac{u\xi}{\tau}(\eta_v - \tau_t) - \frac{1}{\tau^2}\big(\eta^2 - 2u\xi\eta + u^2\xi^2\big)\tau_v - u\xi_t \\ &= C_x + uB_x + u^2A_x + \Big[\frac{u^2}{2} + \frac{1}{\tau}(\eta - u\xi)\Big](B + 2uA) + u\big(C_v + uB_v + u^2A_v\big) - u\big(C + uB + u^2A\big) \\ &\quad - \Big\{\frac{C + uB + u^2A}{\tau} - \frac{\xi}{\tau}\Big[\frac{u^2}{2} + \frac{1}{\tau}(\eta - u\xi)\Big]\Big\}(\tau_x + u\tau_v) - \Big[\frac{u^2}{2} + \frac{1}{\tau}(\eta - u\xi)\Big]\xi_x - \frac{u^3}{2}\xi_v \end{aligned} \quad (8)$$

with the functions $C = \eta_x - \frac{\eta}{\tau}\tau_x$, $B = \eta_v - \xi_x + \frac{\xi}{\tau}\tau_x - \frac{\eta}{\tau}\tau_v$ and $A = \frac{\xi}{\tau}\tau_v - \xi_v$. Since the function u is arbitrary from formula (8), we obtain the following four equations from the coefficients of u powers,

$$\begin{aligned} &A_v - \frac{\tau_v}{\tau}A + \frac{A}{2} = 0, \quad A_x + B_v - \frac{2\xi}{\tau}A - \frac{A}{\tau}\tau_x + \frac{\xi}{2\tau}\tau_x - \frac{B}{\tau}\tau_v - \frac{\xi_x}{2} - \frac{B}{2} = 0, \\ &\frac{\xi}{\tau}(\eta_v - \tau_t) - \frac{2\xi\eta}{\tau^2}\tau_v + \xi_t + B_x + C_v + \frac{2\eta}{\tau}A - \frac{\xi}{\tau}B - \frac{\xi^2}{\tau^2}\tau_x - \frac{B}{\tau}\tau_x - \frac{\tau_v}{\tau}C + \frac{\xi\eta}{\tau^2}\tau_v + \frac{\xi}{\tau}\xi_x - C = 0, \\ &\eta_t + \frac{\eta}{\tau}(\eta_v - \tau_t) - \frac{\eta^2}{\tau^2}\tau_v - C_x - \frac{\eta}{\tau}B + \frac{C}{\tau}\tau_x - \frac{\xi\eta}{\tau^2}\tau_x + \frac{\eta}{\tau}\xi_x = 0. \end{aligned} \quad (9)$$

Now we have to get the functions τ, ξ, ϕ and η from (9). Although the equations (9) are too complicated to be solved in general, some special solutions can be obtained. Choosing $\tau = 1$ and $\xi = \xi(t)$, we have $A = 0$, $B = \eta_v$ and $C = \eta_x$. If we substitute this into (9), it follows that

$$\eta = 2\alpha\exp(\frac{v}{2}) + ax + c \text{ and } \xi = at + b, \quad (10)$$

where the function α satisfies the heat equation $\alpha_t = \alpha_{xx}$, and a, b, c are arbitrary constants. From (10) we obtain the nonclassical potential symmetry infinitesimal generator

$$V_0 = \frac{\partial}{\partial t} + (at + b)\frac{\partial}{\partial x} + \Big[(2\alpha_x + u\alpha)\exp(\frac{v}{2}) + a\Big]\frac{\partial}{\partial u} + \Big[2\alpha\exp(\frac{v}{2}) + ax + c\Big]\frac{\partial}{\partial v}. \quad (11)$$

As is easy to see from (11), we do not obtain any new symmetries different from the Lie symmetries of the adjoint equation (3). Therefore no new explicit solutions can be obtained by using the symmetry generator (11).

Choosing the functions $\tau = 1$ and $\xi = \xi(x)$, we have $A = 0$, $B = \eta_v - \xi_x$ and $C = \eta_x$. Substituting this into (9), we find

$$\eta = 2\alpha_1(x,t)\exp(\frac{v}{2}) + \beta_1(x,t), \tag{12}$$

and

$$\begin{aligned} &\xi_{xx} - 2\xi\xi_x + \beta_{1x} = 0,\\ &2(\alpha_{1t} - \alpha_{1xx} + 2\alpha_1\xi_x)\exp(\frac{v}{2}) = 0,\\ &\beta_{1t} - \beta_{1xx} + 2\beta_1\xi_x = 0. \end{aligned} \tag{13}$$

Combination of the first and third equation of (13) leads to

$$\begin{aligned} &\beta_1 = \xi^2 - \xi_x + k_1,\\ &\xi_{xxx} - 2\xi\xi_{xx} - 4\xi_x^2 + 2\xi^2\xi_x + 2k_1\xi_x = 0, \end{aligned} \tag{14}$$

where k_1 is a constant. From (14) we can derive the following general solutions:

$$\begin{aligned} &\xi_1 = a_0 + a_1\psi(x),\ \beta_{11} = 2a_0a_1\psi(x),\ \alpha_{11} = \pm\beta_{11},\\ &\xi_2 = 3k\psi(x),\ \beta_{12} = k^2[k_1^2 + 3\psi^2(x)],\ \alpha_{12} = \pm\beta_{12},\\ &\xi_3 = k\psi(x),\ \beta_{13} = 0,\ \alpha_{13} = \pm\xi_3, \end{aligned} \tag{15}$$

where $a_0 = \pm\sqrt{k_1^2k^2 - C}$, $a_1 = k$, $\psi(x) = \pm k_1 \tan[\pm k_1(kx + c_0)]$, C, and c_0 are suitable constants, and

$$\begin{aligned} &\xi_4 = a_2 + a_3\phi(x),\ \beta_{14} = 2a_2a_3\phi(x),\ \alpha_{14} = \pm\beta_{14},\\ &\xi_5 = -3k\phi(x), \beta_{15} = k^2[3\phi^2(x) - k_1^2], \alpha_{15} = \pm\beta_{15},\\ &\xi_6 = k\phi(x),\ \beta_{16} = 0,\ \alpha_{13} = \pm\xi_6, \end{aligned} \tag{16}$$

where $a_2 = \pm\sqrt{-k_1^2k^2 - C_0}$, $a_3 = -k$, $\phi(x) = \pm k_1 \coth[\pm k_1(kx + c_1)]$, C_0, and c_1 are suitable constants. The expressions ξ_3, ξ_6 were given as p_2 in [8], the other expressions do not seem to have been found in previous work. From (15) and (16), many nonclassical potential symmetry generators and exact solutions can be obtained. In what follows, we only consider the following three typical cases.

Case I. Setting the functions $\xi = 1 + \tan x$ and $\beta = \alpha = 2\tan x$, we obtain the infinitesimal generator

$$\begin{aligned} V_1 = &\frac{\partial}{\partial t} + (1 + \tan x)\frac{\partial}{\partial x} + 2\tan x\Big[2\exp(\frac{v}{2}) + 1\Big]\frac{\partial}{\partial v}\\ &+ \Big\{2\big[u\tan x + 2\sec^2 x\big]\exp(\frac{v}{2})\\ &\quad + \sec^2 x(2 - u)\Big\}\frac{\partial}{\partial u}. \end{aligned} \tag{17}$$

The generator (17) leads to the following characteristic equation:

$$\frac{\mathrm{d}t}{1} = \frac{\mathrm{d}x}{1 + \tan x} = \frac{\mathrm{d}v}{2\tan x[2\exp(\frac{v}{2}) + 1]}. \tag{18}$$

Solving (18), we have

$$\begin{aligned} &\zeta = t - \frac{1}{2}[x + \ln(\sin x + \cos x)],\\ &f(\zeta) = x - \ln(\sin x + \cos x) + 2\ln[2 + \exp(-\frac{v}{2})], \end{aligned} \tag{19}$$

where the function $f(\zeta)$ satisfies

$$f'' + \frac{f'^2}{2} = 2. \tag{20}$$

Solving (20), we obtain the general solution $f(\zeta) = -2\zeta + 2\ln(c_2 - c_1\exp 2\zeta) - 4\ln 2$. If we substitute this into the second equation of (19), we arrive at

$$\begin{aligned} &2t - 2\ln(\sin x + \cos x) + 2\ln\Big[2 + \exp(-\frac{v}{2})\Big]\\ &-2\ln\big\{c_2 - c_1\exp(2t - [x + \ln(\sin x + \cos x)])\big\}\\ &+ 4\ln 2 = 0. \end{aligned} \tag{21}$$

From (21) we calculate the derivative of v with respect to x:

$$u(x,t) = -2\frac{\frac{c_2(\cos x - \sin x)}{4} + \frac{c_1}{4}\exp(2t - x)}{\frac{c_2(\cos x + \sin x)}{4} - \frac{c_1}{4}\exp(2t - x) - 2\exp(t)}. \tag{22}$$

It is easy to verify that (22) is a new explicit solution of the Burgers equation (1). If we choose $f(\zeta) = 2\ln(\cosh\zeta)$ and substitute it into the second equation of (19), we obtain

$$\begin{aligned} &x - \ln(\sin x + \cos x) + 2\ln[2 + \exp(-\frac{v}{2})]\\ &-2\ln\{\cosh(t - \frac{1}{2}[x + \ln(\sin x + \cos x)])\} = 0. \end{aligned} \tag{23}$$

From (23), the derivative of v with respect to x can be calculated:

$$u(x,t) = -2\frac{\frac{\cos x - \sin x}{2} + \frac{1}{2}\exp(2t-x)}{\frac{\cos x + \sin x}{2} - \frac{1}{2}\exp(2t-x) - 2\exp(t)}. \quad (24)$$

It is easy to show that (24) is also a new explicit solution of the Burgers equation (1). In addition, when choosing $f(\zeta) = \pm 2\zeta$, the following special solution is found

$$u(x,t) = \frac{2}{1-2\exp(x-t)}$$

and

$$u(x,t) = \frac{2(\sin x - \cos x)}{\cos x + \sin x - 2\exp t},$$

respectively.

Case II. With the choice $\xi = \tan x - 1$ and $\beta = \alpha = -2\tan x$, we derive the infinitesimal generator

$$V_2 = \frac{\partial}{\partial t} + (\tan x - 1)\frac{\partial}{\partial x} - 2\tan x\Big[2\exp(\frac{v}{2}) + 1\Big]\frac{\partial}{\partial v} \Big[-2(u\tan x + 2\sec^2 x)\exp\big(\frac{v}{2}\big) - (2+u)\sec^2 x\Big]\frac{\partial}{\partial u}. \quad (25)$$

With the corresponding steps as just described we derive from (25) the following characteristic equation

$$\frac{\mathrm{d}t}{1} = \frac{\mathrm{d}x}{\tan x - 1} = \frac{\mathrm{d}v}{-2\tan x[2\exp(\frac{v}{2}) + 1]}. \quad (26)$$

Solving (26), we have

$$\zeta_1 = t - \frac{1}{2}[-x + \ln(\sin x - \cos x)], \quad f(\zeta_1) = -x - \ln(\sin x - \cos x) + 2\ln\Big[2 + \exp(-\frac{v}{2})\Big], \quad (27)$$

where the function f satisfies (20). Therefore,

$$2t - 2\ln(\sin x - \cos x) + 2\ln\Big[2 + \exp(-\frac{v}{2})\Big] - 2\ln\Big\{c_2 - c_1\exp(2t - [x + \ln(\sin x - \cos x)])\Big\} + 4\ln 2 = 0. \quad (28)$$

From (28), the derivative of v with respect to x is

$$u(x,t) = -2\frac{\frac{-c_2(\cos x + \sin x)}{4} + \frac{c_1}{4}\exp(2t-x)}{\frac{c_2(\cos x - \sin x)}{4} + \frac{c_1}{4}\exp(2t+x) - 2\exp(t)}. \quad (29)$$

Again one can verify that (29) is a new explicit solution of the Burgers equation (1). Similar to case I, if we choose $f(\zeta_1) = 2\ln(\cosh\zeta_1)$ and substitute it into the second equation of (27), we obtain

$$-x - \ln(\sin x - \cos x) + 2\ln[2 + \exp(-\frac{v}{2})] - 2\ln\Big\{\cosh(t - \frac{1}{2}[-x + \ln(\sin x - \cos x)])\Big\} = 0. \quad (30)$$

From (30), we have

$$v(x,t) = -2\ln\Big[\frac{\cos x - \sin x}{2} + \frac{1}{2}\exp(2t+x) - 2\exp(t)\Big] + 2t. \quad (31)$$

From (31), the derivative of v with respect to x is

$$u(x,t) = -2\frac{\frac{-(\cos x - \sin x)}{2} + \frac{1}{2}\exp(2t+x)}{\frac{\cos x - \sin x}{2} + \frac{1}{2}\exp(2t+x) - 2\exp(t)}. \quad (32)$$

Also expression (32) is a new explicit solution of the Burgers equation (1). If we let $f(\zeta_1) = \pm 2\zeta_1$, the following special solutions arise:

$$u(x,t) = \frac{2}{2\exp(-x-t) - 1} \quad \text{and} \quad u(x,t) = \frac{2(\sin x + \cos x)}{\cos x - \sin x + 2\exp t},$$

respectively.

Case III. Finally, with the choice $\xi = 3\tan x$ and $\beta = \alpha = 2(1+3\tan^2 x)$, we find the infinitesimal generator

$$V_3 = \frac{\partial}{\partial t} + (3\tan x)\frac{\partial}{\partial x} + 2(1+3\tan^2 x)\left[2\exp(\frac{v}{2})+1\right]\frac{\partial}{\partial v} + \left\{2[u(1+3\tan^2 x)+12\tan x\sec^2 x]\exp(\frac{v}{2})+3(4\tan x-u)\sec^2 x\right\}\frac{\partial}{\partial u}. \quad (33)$$

From (33), the characteristic equation is

$$\frac{\mathrm{d}t}{1} = \frac{\mathrm{d}x}{3\tan x} = \frac{\mathrm{d}v}{2(1+3\tan^2 x)[2\exp(\frac{v}{2})+1]}. \quad (34)$$

Solving (34), we have

$$\zeta_2 = t - \frac{1}{3}\ln(\sin x), \quad f(\zeta_2) = \frac{2}{3}\ln(\sin x) - 2\ln(\cos x) + 2\ln\left[2+\exp(-\frac{v}{2})\right], \quad (35)$$

where the function $f(\zeta_2)$ this time satisfies

$$f'' + \frac{f'^2}{2} + 5f' = -8. \quad (36)$$

Solving (36), we have $f(\zeta_2) = -8\zeta_2 + 2\ln[c_1 - c_2\exp(3\zeta_2)] - 2\ln 6$, and substituting it into the second equation of (35), we obtain

$$v(x,t) = -2\ln\left[\frac{c_1\sin x\cos x\exp(-3t)}{6} - \frac{c_1\cos x}{6} - 2\exp(-t)\right] + 2t. \quad (37)$$

From (37), the derivative of v with respect to x is

$$u(x,t) = -2\frac{\frac{c_1\cos 2x\exp(-3t)}{6} + \frac{c_1\sin x}{6}}{\frac{c_1\sin x\cos x\exp(-3t)}{6} - \frac{c_1\cos x}{6} - 2\exp(t)}, \quad (38)$$

which again is a new exact solution of (1). In particular, if $f(\zeta_2) = -2\zeta_2$, we derive the following special solution:

$$u(x,t) = \frac{2\sin x}{\cos x - 2\exp t}, \quad u(x,t) = \frac{2\cos 2x}{2\exp(4t) - \cos x\sin x},$$

for $f(\zeta_2) = -8\zeta_2$.

3. Conclusion

In summary, six kinds of new nonclassical potential symmetry generators of the Burgers equation are determined in this paper, and three classes of new explicit solutions are derived by using nonclassical methods to three typical nonclassical potential symmetry generators. These explicit solutions can not be calculated from either one of the Lie symmetry group, nonclassical Lie symmetries of the Burgers equation, its adjoint equation and the Hopf-Cole transformation (which reduces solutions of the Burgers equation to positive solutions of the heat equation).

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